

THERMOPOROELASTIC COUPLING WITH APPLICATION TO CONSOLIDATION

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SUMMARY

Based on a fully coupled thermoporoelastic formulation, this paper discusses the general conditions where the coupling should be maintained, and where a partial or full decoupling technique may be applied. This exercise is aimed at providing practical solutions for the coupled thermoporoelastic analyses where excessive manipulations and unreasonable simplifications are minimized. The necessity for full coupling and the justification for decoupling are illustrated in a thermoporoelastic application of a one-dimensional consolidation scenario. © 1997 by John Wiley & Sons, Ltd.

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INTRODUCTION

As multidisciplinary approach becomes increasingly popular, numerous complex coupled processes can be effectively analysed. One typical coupled phenomenon can be attributed to the thermomechanical response of fluid-saturated media, with applications to such case as the extraction of geothermal energy, enhanced hydrocarbon recovery using thermal flooding, and reservoir stimulation under non-isothermal conditions. It is understood that these circumstances may result in strong coupling between heat flow, interstitial pore fluid flow and solid matrix deformation. Under specific circumstances, the thermoporoelastic processes may degenerate into partially coupled situations where merely thermoelastic or poroelastic effects are manifest. The examples of former cases include the storage of high-level nuclear waste in dry rock masses, contaminant site remediation through hot air stripping, and rock fracturing using non-liquid-type heat sources, while those of latter cases encompass the consolidation in fluid-saturated porous materials, fault dislocation due to fluid withdrawal, and borehole stability in oil/gas production, all in the isothermal environment.

As a result of mathematical difficulties, implementation of a fully coupled formulation has been hindered in many instances where convenient analytical solutions are desired. Under such conditions, alternative numerical approaches have been proved to be effective but at the cost of undue computational expenses and efforts, and frequently are discouraged in engineering applications. Experience shows that the mathematical sophistication cannot be placed as natural

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advantages for the coupled approaches. Sound physical implications and significant influences may be the only reasons to render the coupled solutions. Similarly, one specific mechanism such as the mechanical impact may not always play a leading role in coupled processes. In the thermoelastic or poroelastic analyses, justifications must be made to identify the major factors, and to incorporate important or exclude trivial coupling mechanisms such as chemical processes, heterogeneities and structural abnormalities. In order to provide practical solutions for the coupled analyses while avoiding unreasonable simplifications, this paper examines the conditions where the coupling should be maintained, or where a partial or full decoupling technique may be invoked. This strategic analysis is highlighted in a study of 1-D consolidation scenario where the non-isothermal influences are considered.

A FULLY COUPLED THERMOPOROELASTIC FORMULATION

In general, the ‘coupling’ is interpreted as the ‘simultaneous occurrence of at least two mechanisms in an interactive fashion’. The responses of fluid flow and heat transfer in deformable porous media are demonstrated in a triangular-chain fashion, where the changes in either solid body strain, pore pressure, or material temperature may result in the corresponding unequilibrium of momentum, mass and energy systems. For a linear, quasi-steady elastic, fluid saturated and non-isothermal system, a fully coupled thermoporoelastic formulation¹⁻⁴ may be described as the following.

Conservation of momentum:

$$Gu_{i,jj} + (\lambda + G)u_{k,ki} = \alpha p_{,i} + \beta T_{,i} \quad (1)$$

Conservation of mass:

$$\frac{k}{\mu} p_{,kk} = \alpha \dot{\epsilon}_{kk} - \alpha_h \dot{T} + \alpha^* \dot{p} + Q_f \quad (2)$$

Conservation of energy:

$$K^* T_{,kk} = q_f s T_{,k} + \beta T_0 \dot{\epsilon}_{kk} + s^* \dot{T} + Q_h \quad (3)$$

where a comma stands for differentiation and summation is implied over the repeated Latin subscripts; a superscripted dot identifies the derivative with respect to time t ; u is the displacement, p is the pressure, T is the temperature, ϵ_{kk} is the volumetric strain, λ and G are Lamé’s constants, α is Biot’s coefficient, β is the thermal expansion factor, k is the permeability, μ is the fluid dynamic viscosity, α^* is the lumped compressibility, α_h is the thermal expansion coefficient, K^* is the thermal conductivity, q_f is Darcy’s velocity of fluid flow, s is the intrinsic heat capacity for fluid, s^* is the lumped intrinsic heat capacity, T_0 is the reference temperature, Q_f and Q_h are the internal or external fluid and heat sources, respectively.

More specific definitions for the above parameters include

$$\alpha = 1 - \frac{K}{K_s} \quad (4)$$

where K and K_s are the bulk moduli of solid skeleton and grain,

$$\beta = \alpha_h (3\lambda + 2G) \quad (5)$$

$$\alpha^* = \frac{n}{K_f} + \frac{1-n}{K_s} \quad (6)$$

where n is the porosity, K_f is the fluid bulk modulus, and

$$\alpha_h = n\alpha_f + (1 - n)\alpha_s \quad (7)$$

with α_f and α_s being the thermal expansion coefficients for fluid and solid,

$$K^* = nK_f^* + (1 - n)K_s^* \quad (8)$$

in which K_f^* and K_s^* are the thermal conductivities for fluid and solid,

$$q_f = \frac{-k}{\mu n} p_{,k} \quad (9)$$

$$s = \rho_f c_f n \quad (10)$$

where ρ_f and c_f are the density and heat capacity of fluid,

$$s^* = s + \rho_s c_s (1 - n) \quad (11)$$

with ρ_s and c_s being the density and heat capacity of solid.

In addition to the traditional quasi-steady mechanical equilibrium as well as transient fluid flow and heat transport, the coupling terms embedded in equations (1)–(3) are shown as first two terms on the right-hand side of each equation. They represent, respectively:

Term (1a)— $\alpha p_{,i}$: modification of the effective stress due to the influence of fluid pressure;

Term (1b)— $\beta T_{,i}$: adjustment of total strain as a result of thermal expansion or contraction;

Term (2a)— $\alpha \dot{\epsilon}_{kk}$: mass rate changes expelled by the volumetric strain;

Term (2b)— $\alpha_h \dot{T}$: rate variation as a result of thermal expansion or contraction of fluid and solid;

Term (3a)— $q_f s T_{,k}$: unequilibrium of energy transport due to forced thermal convection in fluid flow;

Term (3b)— $\beta T_0 \dot{\epsilon}_{kk}$: energy lost as a result of solid elastic deformation in the form of thermal expansion or contraction.

Figure 1 represents a synthesis of the above-mentioned interactive thermoporoelastic behaviour. It may be noted that any influence of one mechanism over the other receives a retroactive impact from its counterpart. This intricate interaction provides the convincing evidence for the application of comprehensive thermal–hydraulic–mechanical coupling.

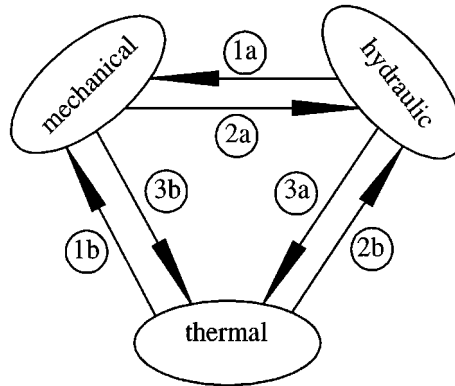


Figure 1. Relationship of thermoporoelastic coupling

PARTIALLY DECOUPLED FORMULATION

For a conserved thermoporoelastic system, the implementation of a fully coupled formulation and solution may enhance computational accuracy and avoid phenomenological misrepresentation. However, maintaining a fully coupled thermoporoelastic formulation excludes, in most cases, the possibility of using convenient analytical approaches. In practice, partially decoupled formulations are developed where the analytical solutions are desired. However, this decoupling requires a justification on the physical intuition, in addition to an analytical quantification.

The partial decoupling is designed to derive the analytical solutions sequentially. It differs from the complete decoupling in its partial replication of one interactive mechanism, instead of ignoring it completely. The decoupling is frequently based on the physical ground, such as described by the following scenarios.

Case 1: For the stiff material or for the material subject to relatively small load and insignificant fluid flow, the effects of elastic deformation and thermal convection may be negligible. The coupled thermoporoelastic formulation is simplified to

$$\frac{k}{\mu} p_{,kk} = -\alpha_h \dot{T} + \alpha^* \dot{p} + Q_f \quad (12)$$

$$K^* T_{,kk} = s^* \dot{T} + Q_h \quad (13)$$

The temperature T can be derived from equation (13), and then substituted into equation (12) to obtain the pressure p .

Case 2: For the stiff material or for the material subject to relatively small load, and negligible influence of thermal expansion or contraction on the rate change of fluid flow, one has

$$\frac{k}{\mu} p_{,kk} = \alpha^* \dot{p} + Q_f \quad (14)$$

$$K^* T_{,kk} = q_f s T_{,k} + s^* \dot{T} + Q_h \quad (15)$$

The pressure p can be obtained from equation (14), and then substituted into equation (15) by applying the Darcy's law in q_f , to determine the temperature T .

Case 3: For less significant influence of rate changes of elastic volumetric deformation on those of both fluid flow and thermal transport, and negligible thermal convection, the thermoporoelastic formulation becomes

$$Gu_{i,jj} + (\lambda + G)u_{k,ki} = \alpha p_{,i} + \beta T_{,i} \quad (16)$$

$$\frac{k}{\mu} p_{,kk} = -\alpha_h \dot{T} + \alpha^* \dot{p} + Q_f \quad (17)$$

$$K^* T_{,kk} = s^* \dot{T} + Q_h \quad (18)$$

Solving T from equation (18) and substituting T into equation (17) to obtain p , \mathbf{u} can be finally derived from equation (16) after substituting both T and p . It should be noted that the decoupled simplification does not eliminate the inclusion of the impact of both pressure and temperature on the change in solid deformation (e.g. equation (16)). This non-retroactive response indeed exists in the real situation, depending on relative magnitudes of unknowns.

Other customized partial decoupling schemes can be implemented by neglecting, based on the pertinent physical ground, any negligible coupling terms in equations (1)–(3).

1-D CONSOLIDATION MODEL

The strategies to implement either full coupling or partial decoupling may be demonstrated through examining a case in the one-dimensional consolidation, frequently referred as ‘a column problem’. As in a traditional soil test,^{5–7} a laterally confined porous column with the height h is situated over a rigid, impermeable, and adiabatic base. A load with magnitude F_0 is applied instantaneously at the top of the column ($x = 0$), forcing the column to consolidate while allowing the fluid to escape and the temperature to dissipate from the top.

By neglecting the forced thermal convection as well as fluid and heat sources, equations (1)–(3) can be written for a 1-D non-isothermal consolidation due to external force as

$$(\lambda + 2G) \frac{\partial^2 u}{\partial x^2} = \alpha \frac{\partial p}{\partial x} + \beta \frac{\partial T}{\partial x} \quad (19)$$

$$\frac{k}{\mu} \frac{\partial^2 p}{\partial x^2} = \alpha^* \frac{\partial p}{\partial t} + \alpha \frac{\partial^2 u}{\partial x \partial t} - \alpha_h \frac{\partial T}{\partial t} \quad (20)$$

$$K^* \frac{\partial^2 T}{\partial x^2} = \beta T_0 \frac{\partial^2 u}{\partial x \partial t} + s^* \frac{\partial T}{\partial t} \quad (21)$$

The boundary and initial conditions can be given as

$$\begin{aligned} \frac{\partial u(0, t)}{\partial x} &= E_0 = -\frac{F_0}{\lambda + 2G}, & p(0, t) &= T(0, t) = 0 \\ u(h, t) &= 0, & \frac{\partial p(h, t)}{\partial x} &= \frac{\partial T(h, t)}{\partial x} = 0 \\ p(x, 0) &= p_0, & T(x, 0) &= T_0 \end{aligned} \quad (22)$$

where F_0 is the external load, P_0 and T_0 are the initial pressure and temperature, respectively.

Integration with respect to x in equation (19), yields

$$\eta \frac{\partial u}{\partial x} = \alpha p + \beta T + f_1 \quad (23)$$

where

$$\eta = \lambda + 2G \quad (24)$$

and the integration constant f_1 can be determined by applying the conditions in equation (22), which gives

$$f_1 = \eta E_0 = -F_0 \quad (25)$$

The spatial derivative of u can be obtained from equation (23). Differentiating this derivative with respect to t , substituting the result into equations (20) and (21), one has

$$\frac{\partial^2 p}{\partial x^2} = g_{11} \frac{\partial p}{\partial t} + g_{12} \frac{\partial T}{\partial t} \quad (26)$$

$$\frac{\partial^2 T}{\partial x^2} = g_{21} \frac{\partial p}{\partial t} + g_{22} \frac{\partial T}{\partial t} \quad (27)$$

where

$$\begin{aligned}
 g_{11} &= \frac{\mu}{k} \left(\alpha^* + \frac{\alpha^2}{\eta} \right) \\
 g_{12} &= \frac{\mu}{k} \left(\frac{\alpha\beta}{\eta} - \alpha_h \right) \\
 g_{21} &= \frac{\alpha\beta T_0}{K^* \eta} \\
 g_{22} &= \frac{1}{K^*} \left(s^* + \frac{\beta^2 T_0}{\eta} \right)
 \end{aligned} \tag{28}$$

The finite Fourier transform can be used, which is defined by

$$\begin{aligned}
 \bar{p}(m, t) &= \int_0^h p(x, t) \sin \zeta x \, dx \\
 p(x, t) &= \frac{2}{\pi} \sum_{m=0}^{\infty} \bar{p}(m, t) \sin \zeta x
 \end{aligned} \tag{29}$$

$$\begin{aligned}
 \bar{T}(m, t) &= \int_0^h T(x, t) \sin \zeta x \, dx \\
 T(x, t) &= \frac{2}{\pi} \sum_{m=0}^{\infty} \bar{T}(m, t) \sin \zeta x \\
 \zeta &= \frac{(2m+1)\pi}{2h}
 \end{aligned} \tag{30}$$

The transformation defined in equation (29) automatically satisfies the boundary conditions described in equation (22). The initial conditions in equation (22) in the Fourier domain can be expressed as

$$\begin{aligned}
 \bar{p}(m, 0) &= p_0 \int_0^h \sin \zeta x \, dx = \frac{p_0}{\zeta} \\
 \bar{T}(m, 0) &= T_0 \int_0^h \sin \zeta x \, dx = \frac{T_0}{\zeta}
 \end{aligned} \tag{31}$$

Using the Fourier transform, equations (26) and (27) can be rewritten as

$$- \zeta^2 \bar{p} = g_{11} \frac{d\bar{p}}{dt} + g_{12} \frac{d\bar{T}}{dt} \tag{32}$$

$$- \zeta^2 \bar{T} = g_{21} \frac{d\bar{p}}{dt} + g_{22} \frac{d\bar{T}}{dt} \tag{33}$$

Solving equations (32) and (33) simultaneously, and applying the initial conditions in equation (31), gives

$$\bar{p} = \frac{1}{\xi} (\phi_{12} e^{-\gamma_{11} \xi^2 t} + \phi_{11} e^{-\gamma_{22} \xi^2 t}) \quad (34)$$

$$\bar{T} = \frac{1}{\xi} (\phi_{22} e^{-\gamma_{11} \xi^2 t} + \phi_{21} e^{-\gamma_{22} \xi^2 t}) \quad (35)$$

where

$$\phi_{ij} = -\frac{\gamma_{ij} \gamma_j^*}{\gamma^*} \quad (i = 1, 2; j = 1, 2) \quad (36)$$

and where

$$\begin{aligned} \gamma_{ij} &= \frac{g_{ij}}{g^*} \quad (i = 1, 2; j = 1, 2) \\ g^* &= g_{11} g_{22} - g_{12} g_{21} \\ \gamma_1^* &= \gamma_{12} T_0 - \gamma_{22} p_0 \\ \gamma_2^* &= \gamma_{21} p_0 - \gamma_{11} T_0 \\ \gamma^* &= \gamma_{11} \gamma_{22} - \gamma_{12} \gamma_{21} \end{aligned} \quad (37)$$

The pressure and temperature can be finally derived by substituting equations (34) and (35) into equation (29), which yields

$$p(x, t) = \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{1}{\xi} (\phi_{12} e^{-\gamma_{11} \xi^2 t} + \phi_{11} e^{-\gamma_{22} \xi^2 t}) \sin \xi x \quad (38)$$

$$T(x, t) = \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{1}{\xi} (\phi_{22} e^{-\gamma_{11} \xi^2 t} + \phi_{21} e^{-\gamma_{22} \xi^2 t}) \sin \xi x \quad (39)$$

The total displacement can be obtained by the integration with respect to the derivative of u in equation (23); then

$$\begin{aligned} u &= \int_0^h \left[\frac{1}{\eta} (\alpha p + \beta T) + E_0 \right] dx \\ &= E_0 h + \frac{2}{\pi \eta} \sum_{m=0}^{\infty} \frac{1}{\xi^2} (\psi_1 e^{-\gamma_{11} \xi^2 t} + \psi_2 e^{-\gamma_{22} \xi^2 t}) \end{aligned} \quad (40)$$

where

$$\begin{aligned} \psi_1 &= \alpha \phi_{12} + \beta \phi_{22} \\ \psi_2 &= \alpha \phi_{11} + \beta \phi_{21} \end{aligned} \quad (41)$$

The general solution of displacement u , however, can be expressed as

$$u(x, t) = E_0(x - h) - \frac{2}{\pi \eta} \sum_{m=0}^{\infty} \frac{1}{\xi^2} (\psi_1 e^{-\gamma_{11} \xi^2 t} + \psi_2 e^{-\gamma_{22} \xi^2 t}) \cos \xi x \quad (42)$$

Equations (38), (39) and (42) represent the solutions of one-dimensional consolidation subject to the non-isothermal environment.

COMPARATIVE ANALYSIS

The comparative analysis is designed to identify the relevant influence for each coupling term described in equations (1)–(3). This work is particularly useful for achieving analytical solutions in which unimportant coupling terms may be neglected. The work may also reflect the need for a fully coupled thermoporoelastic formulation, in order to preserve the maximum flexibility and accuracy in the modelling. The selected parameters, with respect to the hydraulic, thermal, mechanical and geometric categories, are listed in Table I. For generality, the pressure p , temperature T , and displacement u are expressed in a normalized fashion with respect to their maximum values, so is the distance. Similarly, the dimensionless time is defined as

$$\tau = \frac{c_v t}{h^2} \quad (43)$$

where h is the height of the consolidating column, c_v is the coefficient of consolidation and is described as

$$c_v = \frac{k\eta}{\mu} \quad (44)$$

where η is described in equation (24).

Using the given parameters, the results shown in Figure 2 represent an interactive response of dissipating pressure and temperature to the increase of displacement due to the column consolidation. In comparison with the diffusive temperature dissipation, the change in

Table I. Selected thermoporoelastic parameters

Category	Symbol	Value	Unit
Hydraulic	k	10^{-9}	m^2
	μ	20	kg/(m h)
	n	0.2	Dimensionless
	p_0	10^4	Pa
Thermal	α_f	10^{-5}	$1/^\circ\text{C}$
	α_s	10^{-6}	$1/^\circ\text{C}$
	K_f^*	1000	J/(m h $^\circ\text{C}$)
	K_s^*	100	J/(m h $^\circ\text{C}$)
	ρ_f	1000	kg/m ³
	ρ_s	2000	kg/m ³
	c_f	500	J/(kg $^\circ\text{C}$)
	c_s	200	J/(kg $^\circ\text{C}$)
	T_0	100	$^\circ\text{C}$
Mechanical	α	0.9	Dimensionless
	v	0.25	Dimensionless
	E	5×10^9	Pa
	K_f	5×10^9	Pa
	K	2×10^9	Pa
	K_s	2×10^{10}	Pa
	F_0	10^8	Pa
Geometric	h	100	m
	x	5	m

the pressure appears to be earlier with slight slope perturbation as a result of thermoporoelastic response. In contrast, the displacement reveals a delayed increase after a sluggish preconsolidation.

The dramatic changes are identified in the pressure analysis, in which the solutions vary from full decoupling to complete coupling (Figure 3). The solution with total decoupling replicates a purely diffusive fluid flow where isothermal and non-mechanical environments prevail. The solution with total coupling maintains all coupled terms as indicated in equations (1)–(3) except the thermal convection. At later stage, these two solutions appear to differ noticeably in dimensionless time cycles, where the thermoporoelastic coupling seems to create substantial impedance to the pressure dissipation, in addition to a slope perturbation, for the latter solution. The partial decoupling, as shown in Figure 3, introduces some interesting results, which are either radical (less uniform) when the deformation is decoupled, or intermediate (between full decoupling and coupling) when the temperature is decoupled. The former case indicates that the delay of pressure development can be greatly reduced while the influence of solid deformation is discarded; conversely, the latter case implies that the temperature variation may become an important catalyst in preventing an early pressure dissipation. As a result, a complete coupling appears to be critical to achieve reasonably accurate results for the pressure analysis.

For the present conditions and assumptions, the partial decoupling seems to be justifiable for the analyses of temperature (Figure 4), as well as displacement (Figure 5). In all cases, the discrepancies are negligibly small, even after the modelling parameters are drastically modified. Therefore, the variations in temperature and deformation are less likely affected by other interacting mechanisms. In reality, sequential solutions can be readily obtained through initially solving a decoupled thermal transport equation. However, it should be cautioned that the results can be dramatically modified if some conditions and assumptions are altered (e.g. adding fluid and heat sources or considering the effect of thermal convection).

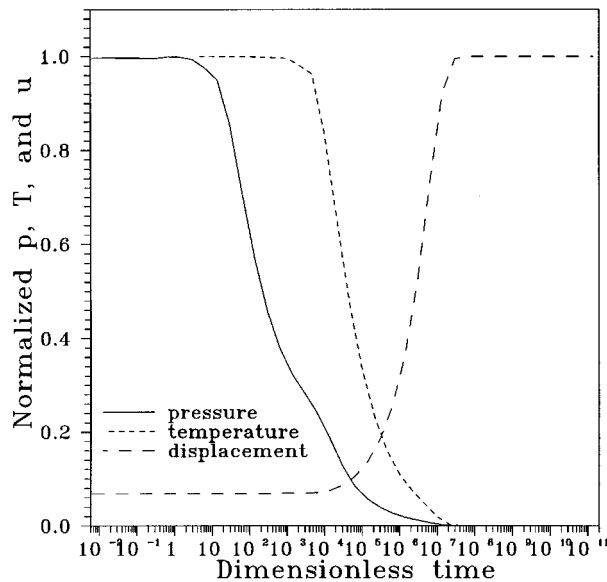


Figure 2. Thermoporoelastic evolution

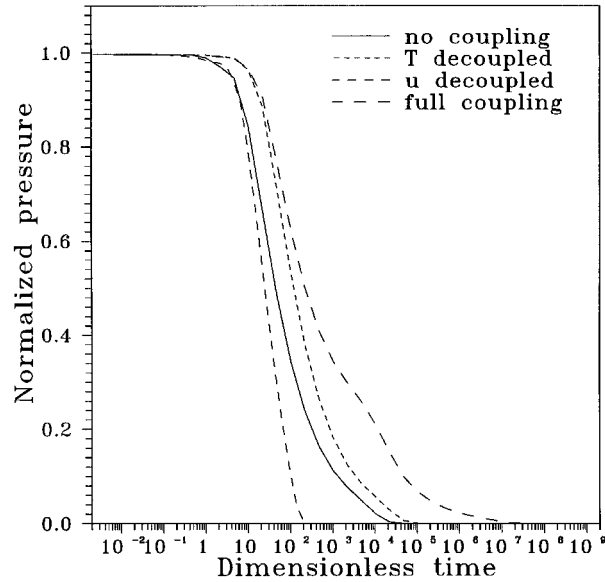


Figure 3. Temporal pressures under different coupling schemes

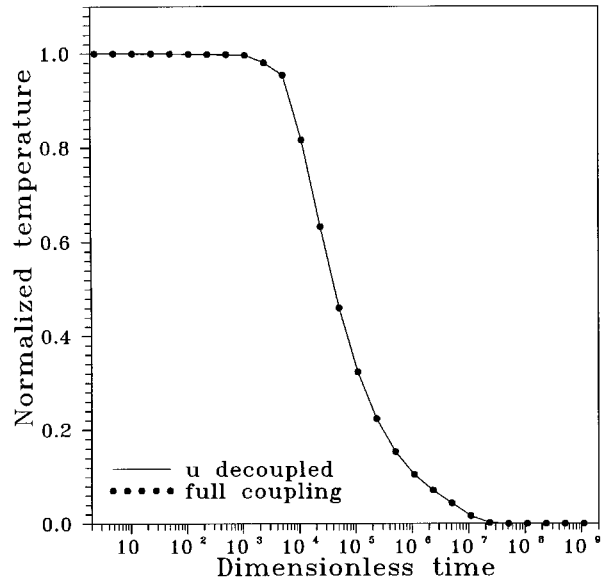


Figure 4. Temporal temperatures under different coupling schemes

For the spatial pressure variation along the consolidating column, Figure 6 illustrates a similar trend of pressure distribution in terms of various coupling schemes when the dimensionless time $\tau = 1000$, as compared to the temporal pressure changes. The result for the temperature decoupling is still sandwiched between those subject to full coupling and complete decoupling. The

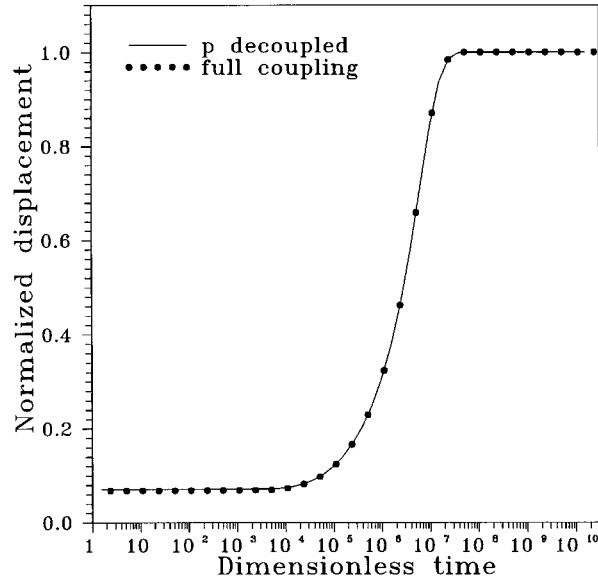


Figure 5. Temporal displacements under different coupling schemes

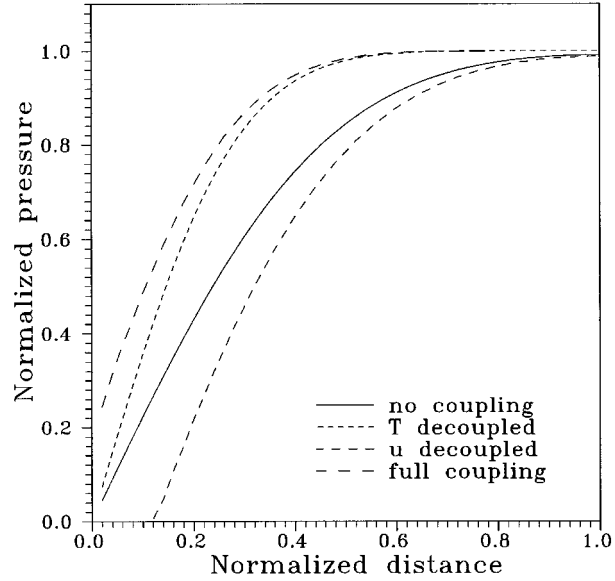


Figure 6. Spatial pressures under different coupling schemes

pressure change, without restricted by the deformation, represents the most conservative pressure changes, as corresponding to the comparatively remote pressure variation from the external load, as shown in Figure 6. Consistently, the result under the full coupling reaches the comparatively maximal magnitude throughout the consolidating column.

CONCLUSIONS

A comprehensive thermoporoelastic formulation is presented to elicit a fact that the development of a fully coupled thermal–hydraulic–mechanical approach is necessary to minimize potential errors while modelling the behaviour of poroelastic media under non-isothermal conditions. However, the full coupling may not be feasible if the engineering-oriented analytical solutions are desired. As a result, simplification, such as using partial decoupling, should be attempted when the influence of each coupling term is determined and the omission of such term is justified on physical and analytical ground. The analyses of various coupling, partial and full decoupling schemes along with their applicable circumstances are demonstrated in the modelling of a 1-D consolidation scenario subject to the non-isothermal environment.

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